Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 207 (2006) 599–616

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

On the mean oscillation of the Hessian of solutions to the Monge–Ampère equation

Qingbo Huang [1]

Department of Mathematics & Statistics, Wright State University, Dayton, OH 45435, USA

Received 21 September 2004; accepted 15 December 2005

Available online 19 January 2006

Communicated by Luis Caffarelli

Abstract

We give interior a priori estimates for the mean oscillation of second derivatives of solutions to the Monge–Ampère equation $\det D^2u = f(x)$ with zero boundary values, where $f(x)$ is a non-Dini continuous function. If the modulus of continuity of $f(x)$ is $\varphi(r)$ such that $\lim_{r\to 0}\varphi(r)\log(1/r) = 0$, then $D^2u \in$ VMO.

MSC: primary 35J60; secondary 35B65

Keywords: Monge–Ampère equation; Mean oscillation of Hessian; Regularity

1. Introduction

This paper is concerned with the regularity of weak solutions to the Monge–Ampère equation

$$\det D^2u = f(x) > 0, \tag{1.1}$$

where $x \in \Omega$ and Ω is a bounded convex domain in $\mathbb{R}^n$. The regularity of solutions for the Monge–Ampère equation has been extensively studied by many authors. For instance, see [1, 2,5,6,11–16], and references therein. Some historic development of the topic can be found in [5,15]. If Ω is a C^∞ strictly convex domain, $f \in C^3$, and $u|_{\partial\Omega} = g$ with $g \in C^4$, [5] established

E-mail address: qhuang@math.wright.edu.

[1] The author was partially supported by NSF grant No. DMS-0201599.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2005.12.005

interior and boundary $C^{2,\alpha}$ estimates. By [1], if the boundary value g is $C^{1,\alpha}$ and $\partial\Omega$ is $C^{1,\alpha}$ with $\alpha > 1-2/n$, the study of interior estimates is reduced to that on cross sections (or level sets) of the graph of u where the boundary value is affine. By the interior $W^{2,p}$ estimates and Schauder estimates in [2], if $u|_{\partial\Omega}=0$ and f is continuous, then $D^2u \in L^p$ for any $1 \leqslant p < \infty$; if $u|_{\partial\Omega}=0$ and $f \in C^\alpha$, then $D^2u \in C^\alpha$. [16] gave some examples showing that if f is only strictly positive and bounded then u might not be in $W^{2,p}$ and that if f is continuous then D^2u might not be bounded. In the case that f is Dini continuous and $u|_{\partial\Omega}=0$, [16] proved that D^2u is bounded and therefore (1.1) becomes uniformly elliptic. Then by [10] or [4], D^2u is continuous.

Our purpose in this paper is to estimate the mean oscillation of second derivatives of solutions to (1.1) with non-Dini continuous $f(x)$. As a consequence of our result, if f has the modulus of continuity φ (i.e., $|f(x)-f(y)| \leqslant \varphi(|x-y|)$ for $x, y \in \Omega$) such that $\lim_{r\to 0}\varphi(r)\log(1/r)=0$, then $D^2u \in \mathrm{VMO}$, where VMO is the closure of C^∞ in BMO.

We recall spaces $\mathrm{BMO}_\psi(\Omega)$ before stating the main result. Let ψ be a nondecreasing continuous function on $[0,\infty)$ such that $\psi(t)>0$ for $t>0$ and $t/\psi(t)$ is almost increasing which means $t/\psi(t) \leqslant Ks/\psi(s)$ for $0<t<s$. For $g(x) \in L^1(\Omega)$, the mean oscillation of $g(x)$ over $B_r(x_0)$ is given by

$$\operatorname{mosc}_{B_r(x_0)} g = \fint_{B_r(x_0)\cap\Omega} \left|g(x) - g_{x_0,r}\right| dx,$$

where $B_r(x_0)$ is the ball centered at x_0 with radius r, $\fint_A g\,dx$ denotes the average of g over A, and $g_{x_0,r}$ the average of g over $B_r(x_0)\cap\Omega$. For simplicity, set $B_r = B_r(0)$.

A function $g(x) \in L^1(\Omega)$ belongs to $\mathrm{BMO}_\psi(\Omega)$ if there exists a constant C such that

$$\operatorname{mosc}_{B_r(x_0)} g \leqslant C\psi(r),$$

for all $x_0 \in \Omega$, $0 < r \leqslant d = \operatorname{diam}(\Omega)$. Here $\operatorname{diam}(\Omega)$ is the diameter of Ω. We note that $g(x) \in \mathrm{VMO}(\Omega)$ if and only if as $r \to 0$, $\operatorname{mosc}_{B_r(x_0)} g$ converges to 0 uniformly in $x_0 \in \Omega$. It is well known that BMO, VMO, and BMO_ψ are important in many aspects of analysis and PDEs. For further properties of $\mathrm{BMO}_\psi(\Omega)$, see [4] and references therein.

The following is the main result of this paper.

Theorem A. *Let u be a convex solution to* (1.1) *in Ω and $u=0$ on $\partial\Omega$. Let $0<\alpha_n<1$ and $B_{\alpha_n} \subset \Omega \subset B_1$ be a convex domain. Assume that there are constants λ, $\Lambda>0$ such that $\lambda \leqslant f(x) \leqslant \Lambda$ in Ω. Suppose that $f \in C(\overline{\Omega})$ with the modulus of continuity φ such that $t^\gamma/\varphi(t)$ is almost increasing for some $0<\gamma<1$. Then we have the following.*

(i) *For small $\tau > 0$, let $\Phi(r) = \int_r^2 \frac{\varphi(\tau t)}{t}\,dt$. If there exists a constant $C>0$ dependent only on n, λ, Λ, γ, φ but not τ such that $\psi(r) = e^{C\Phi(r)}\varphi(e^{C\Phi(r)}r)$ is increasing and satisfies $\lim_{r\to 0}\psi(r)=0$, then $D^2u \in \mathrm{BMO}_\psi(\Omega')$ for any $\Omega' \Subset \Omega$.*

(ii) *If $\lim_{r\to 0}\varphi(r)\log(1/r)=0$, then $D^2u \in \mathrm{BMO}_{\log^{-(1-\varepsilon)}(1/r)}(\Omega')$ for any $0<\varepsilon<1$ and $\Omega' \Subset \Omega$, and hence $D^2u \in \mathrm{VMO}_{\mathrm{loc}}(\Omega)$.*

We point out that if φ satisfies the Dini condition $\int_0^2 \frac{\varphi(t)}{t}\,dt < \infty$, then $\psi(r)$ is actually $\varphi(r)$. If φ fails to satisfy the Dini condition, by imposing further reasonable assumption on φ, the expression of ψ can be simplified. See Remark 3.2.

We will modify the techniques in [4] to prove the above theorem. There are two obstacles we need to remove. First, the Monge–Ampère equation is not uniformly elliptic and the interior smoothness of solutions relies on $C^{1,\alpha}$ norm of boundary data with $\alpha > 1 - 2/n$ and it hints that the role of Euclidean balls should be replaced by that of cross sections. We need to derive sharp estimates on the eccentricity of sections. Second, the mean oscillation of D^2u is not affine invariant. We should find and use another quantity taking the eccentricity into accounted.

The organization of the rest of the paper is as follows. In Section 2, we discuss the eccentricity of cross sections if $f(x)$ is non-Dini continuous. In Section 3, estimates of the mean oscillation of Hessian of solutions are derived.

For simplicity, from now on, we assume that u is smooth. But all estimates are independent of the smoothness of u and remain valid for weak solutions through appropriate approximation.

2. Eccentricity of cross sections

In this section, our goal is to carefully investigate the eccentricity of sections in terms of $\Phi(r) = \int_r^2 \frac{\varphi(\tau t)}{t}\,dt$, where $\tau > 0$ is a small constant to be determined later and φ is the modulus of continuity of $f(x)$.

Let $u(x)$ be a convex solution of (1.1) in Ω and $u = 0$ on $\partial\Omega$. For $x_0 \in \Omega$, recall the section $S_h(x_0) = S_h(u, x_0)$ for $h > 0$ is defined by

$$S_h(x_0) = \{x \in \Omega\colon\ u(x) \leqslant \ell_{x_0}(x) + h\},$$

where $\ell_{x_0}(x) = u(x_0) + Du(x_0)(x - x_0)$ is the supporting hyperplane of u at x_0.

Let us recall some facts about sections. By [1,7] for $x_0 \in \Omega'$ and $\Omega' \Subset \Omega$, there exists $h_0 = h_0(\Omega')$ such that

$$S_h(x_0) \Subset \Omega \quad \text{for } h \leqslant h_0.$$

Moreover, $\operatorname{diam}(S_h(x_0)) \to 0$ as $h \to 0$.

Since f is positive continuous, from (1.1), for $\varepsilon > 0$, there exists $h_0 > 0$ such that

$$(1-\varepsilon) f(x_0) \leqslant \det D^2u \leqslant (1+\varepsilon) f(x_0) \quad \text{in } S_h(x_0),$$

for $x_0 \in \Omega'$, $0 < h \leqslant h_0$.

We can normalize (or rescale) u and $S_h(x_0)$ in the following way. From Fritz John's Lemma, there exists an ellipsoid E centered at z_0 such that

$$\frac{1}{n} E \subset S_h(x_0) \subset E.$$

Let T be the invertible affine transformation given by $Tx = A(x - z_0)$ satisfying $TE = B_1$ and $Tz_0 = 0$. Set

$$u^*(y) = \frac{1}{C_T}\left[(u - \ell_{x_0})\left(T^{-1}y\right) - h\right], \tag{2.1}$$

where $C_T = (f(x_0)|\det A|^{-2})^{1/n}$. Simple calculation gives rise to

$$(1-\varepsilon) \leqslant \det D^2u^* \leqslant (1+\varepsilon) \quad \text{in } S_h^* = TS_h(x_0).$$

Obviously, $u^* = 0$ on ∂S_h^*. By the theory of the elliptic Monge–Ampère equation

$$h^n \approx \left|S_h(x_0)\right|^2 \approx |E|^2 \approx \left|\det A^{-1}\right|^2, \tag{2.2}$$

and hence $C_T \approx h$. Here and throughout the paper, we use the symbol $a \approx b$ to denote that the quantity a/b is bounded by two positive universal constants from above and below. $a \ll b$ denotes a/b is bounded by a universal constant much smaller than 1. We use C to denote universal constants dependent only on structure constants. For $a \in \mathbb{R}^1$ and $E \subset \mathbb{R}^n$, let $aE = \{ax\colon\ x \in E\}$.

We now prove several lemmas for the normalized solution u^*.

Lemma 2.1. *Let u be a strictly convex function in Ω and satisfy*

$$1-\varepsilon \leqslant \det D^2u \leqslant 1+\varepsilon \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$ is a normalized convex domain. Then there exist $z_0 \in \Omega$ and a linear transformation A such that

$$\det A = 1, \tag{2.3}$$
$$C^{-1} \leqslant \|Ax\| \leqslant C \quad \text{for } \|x\| = 1, \tag{2.4}$$

and for small $\mu > 0$ with $\varepsilon \ll \mu$

$$\left[1 - C(\sqrt{\mu} + \varepsilon/\mu)\right]B_{\sqrt{2}} \subset \sqrt{\mu}^{-1} T S_\mu(u) \subset \left[1 + C(\sqrt{\mu} + \varepsilon/\mu)\right]B_{\sqrt{2}}, \tag{2.5}$$

where $Tx = A(x - z_0)$ and $S_\mu(u,\Omega) = S_\mu(u) = \{x \in \Omega\colon\ u(x) \leqslant \min_\Omega u(x) + \mu\}$.

Proof. Let $w(x)$ be the smooth convex solution to the equation

$$\det D^2 w = 1 \quad \text{in } \Omega,$$

with the boundary value $w = 0$ on $\partial\Omega$. By the comparison principle we get

$$(1+\varepsilon)w \leqslant u \leqslant (1-\varepsilon)w \quad \text{in } \Omega.$$

By the maximum principle, Pogorelov C^2 estimate, regularity theory of fully nonlinear equations, one obtains interior C^∞ estimate for w.

Let $u(x_0) = \min_\Omega u$, $w(z_0) = \min_\Omega w$, and $S_\mu(w, z_0) = S_\mu(w)$.

Now compare $S_\mu(u)$ with $S_\mu(w)$ for small $\mu > 0$. Recall $|\min_\Omega u| \approx \text{const}$. Obviously, $(1+\varepsilon)\min_\Omega w \leqslant \min_\Omega u \leqslant (1-\varepsilon)\min_\Omega w$. Therefore, there exists $C > 0$ such that

$$S_{\mu - C\varepsilon}(w) \subset S_\mu(u) \subset S_{\mu + C\varepsilon}(w). \tag{2.6}$$

We now claim that for $\delta \ll \mu$

$$\partial S_{\mu\pm\delta}(w) \subset N_{C\delta/\sqrt{\mu}}\big(\partial S_\mu(w)\big), \tag{2.7}$$

where N_δ is the δ-neighborhood with respect to the Euclidean distance.

To prove (2.7), let $x \in \partial S_{\mu+\delta}(w)$. Let x_1 be the intersecting point of $\partial S_\mu(w)$ and the segment between z_0 and x. Because w is smooth, $B_{C_1\sqrt{\mu}}(z_0) \subset S_\mu(w) \subset B_{C_2\sqrt{\mu}}(z_0)$. One obtains

$$\delta = \left|w(x) - w(x_1)\right| = |Dw| \cdot |x - x_1| \approx C\sqrt{\mu}|x - x_1|.$$

It yields that $|x - x_1| \leqslant C\delta/\sqrt{\mu}$, and (2.7) follows.

We next compare $S_\mu(w)$ with ellipsoids and claim that

$$\partial S_\mu(w) - z_0 \subset N_{C\mu}(\partial\sqrt{\mu}E), \tag{2.8}$$

where $E = \{x\colon \frac{1}{2}D_{ij}w(z_0)x_i x_j \leqslant 1\}$ and $A - z_0 = \{x - z_0\colon x \in A\}$.

To prove (2.8), it is equivalent to show

$$\partial S_\mu(w) - z_0 \subset (1 + C\sqrt{\mu})\sqrt{\mu}E - (1 - C\sqrt{\mu})\sqrt{\mu}E.$$

If $x - z_0 \in \partial((1 + C\sqrt{\mu})\sqrt{\mu}E)$ and $z_0 = (z_{01}, \dots, z_{0n})$, then by the Taylor formula

$$\begin{aligned} w(x) - w(z_0) &= \frac{1}{2}D_{ij}w(z_0)(x_i - z_{0i})(x_j - z_{0j}) + O\big(\left|D^3w\right||x - z_0|^3\big) \\ &\geqslant \big[(1 + C\sqrt{\mu})\sqrt{\mu}\big]^2 - K|x - z_0|^3, \end{aligned} \tag{2.9}$$

where K is a constant proportional to the bounds of D^3w.

If $C > K$ and $C\sqrt{\mu} \ll 1$, then $w(x) - w(z_0) > (1 + C\sqrt{\mu})\mu - K\mu^{3/2} > \mu$. It can be shown similarly that $(1 - C\sqrt{\mu})\sqrt{\mu}E$ is contained inside $S_\mu(w) - z_0$. Thus we complete the proof of (2.8).

From (2.6)–(2.8), we obtain

$$\partial S_\mu(u) \subset z_0 + N_{C(\mu+\varepsilon/\sqrt{\mu})}(\partial\sqrt{\mu}E). \tag{2.10}$$

Now find the transformation A. Since D^2w is positively definite, we can write $D^2w(z_0) = A^t \cdot A$, where A is the composition of rotation and dilation. Let $Tx = A(x - z_0)$. It is easy to verify that (2.3), (2.4) hold. (2.5) follows from (2.10). The proof of Lemma 2.1 is completed. □

If the shape of Ω is close to that of the ball $B_{\sqrt{2}}$, then one can get better estimates for A and $S_\mu(u)$.

Lemma 2.2. *Let u be a strictly convex function in Ω and satisfy*

$$1 - \varepsilon \leqslant \det D^2u \leqslant 1 + \varepsilon \quad \textit{in } \Omega,$$

$$u = 0 \quad \textit{on } \partial\Omega.$$

Assume that Ω is a convex domain and $(1 - \delta)B_{\sqrt{2}} \subset \Omega \subset (1 + \delta)B_{\sqrt{2}}$. Then there exist $z_0 \in \Omega$ and a linear transformation A such that

$$\det A = 1, \tag{2.11}$$

$$1 - C\delta \leqslant \|Ax\| \leqslant 1 + C\delta \quad \textit{for } \|x\| = 1, \tag{2.12}$$

and for small $\mu > 0$ *with* $\varepsilon \ll \mu$

$$\big[1 - C(\delta\sqrt{\mu} + \varepsilon/\mu)\big]B_{\sqrt{2}} \subset \sqrt{\mu}^{-1} T S_\mu(u) \subset \big[1 + C(\delta\sqrt{\mu} + \varepsilon/\mu)\big]B_{\sqrt{2}}, \tag{2.13}$$

where $Tx = A(x - z_0)$.

Proof. Let w be the convex solution to

$$\det D^2 w = 1 \quad \text{in } \Omega, \tag{2.14}$$

and $w = 0$ on $\partial\Omega$. By the comparison principle

$$(1+\varepsilon)w \leqslant u \leqslant (1-\varepsilon)w, \quad \text{in } \Omega.$$

From the maximum principle and Pogorelov estimate, interior estimates for higher order derivatives of w follow.

Let $P = \frac{1}{2}|x|^2 - 1$. Obviously, the functions $P \pm 3\delta$ are also solutions to (2.14), and since $\partial\Omega \subset N_{\delta\sqrt{2}}(\partial B_{\sqrt{2}})$, $P - 3\delta \leqslant 0 \leqslant P + 3\delta$ on $\partial\Omega$. By the comparison principle

$$-3\delta \leqslant w - \left(\frac{1}{2}|x|^2 - 1\right) \leqslant 3\delta \quad \text{in } \Omega.$$

Since w is smooth, $v = w - P$ satisfies the following uniformly elliptic linear equation

$$\operatorname{tr}\big(D(x)D^2 v\big) = 0,$$

where $D(x) = \int_0^1 \det(\theta D^2 w + (1-\theta)I)(\theta D^2 w + (1-\theta)I)^{-1}\, d\theta$, and I is the $n \times n$ unit matrix. By interior Schauder estimates

$$\|w - P\|_{C^2_{\mathrm{loc}}} \leqslant C\|w - P\|_{L^\infty} \leqslant C\delta.$$

In particular, $|D^2 w(z_0) - I| \leqslant C\delta$, where $w(z_0) = \min_\Omega w$. By differentiating (2.14), we obtain that the function $D(w - P)$ satisfies the linearized equation

$$\operatorname{tr}\big(\big(D^2 w\big)^{-1} D^2\big[D(w - P)\big]\big) = 0.$$

Again by the interior Schauder estimates

$$\big\|D^3(w - P)\big\|_{L^\infty_{\mathrm{loc}}} \leqslant C\big\|D(w - P)\big\|_{L^\infty_{\mathrm{loc}}} \leqslant C\delta.$$

Therefore $\|D^3 w\|_{L^\infty_{\mathrm{loc}}} \leqslant C\delta$. Similar to (2.8), by (2.9) and noting that in current case K can be chosen as $C\delta$, we have

$$\partial S_\mu(w) - z_0 \subset N_{C\delta\mu}(\partial\sqrt{\mu}E),$$

where $E = \{x\colon \frac{1}{2} D_{ij} w(z_0) x_i x_j \leqslant 1\}$. Similar to (2.10), we have

$$\partial S_\mu(u) \subset z_0 + N_{C(\delta\mu+\varepsilon/\sqrt{\mu})}(\partial\sqrt{\mu}E). \tag{2.15}$$

Let $D^2 w(z_0) = A^t A$ and $Tx = A(x - z_0)$. Obviously, (2.11) holds and (2.12) follows from the following estimate

$$(1 - C\delta)|x|^2 \leqslant x^t \cdot D^2 w(z_0) \cdot x = \|Ax\|^2 \leqslant (1 + C\delta)|x|^2.$$

It is easy to show that (2.13) follows from (2.15). So Lemma 2.2 is proved. $\square$

We apply Lemmas 2.1 and 2.2 to get the following estimates.

Lemma 2.3. *Suppose that the assumptions in Lemma* 2.1 *hold and further assume that there is a sequence* $\{\varepsilon_k\}_{k=0}^\infty$ *with* $0 < \varepsilon_{k+1} \leqslant \varepsilon_k$ *and* $\varepsilon_0 = \varepsilon \ll \mu$ *such that for* $k \geqslant 1$

$$1 - \varepsilon_k \leqslant \det D^2 u \leqslant 1 + \varepsilon_k \quad \textit{in } S_{\mu^k}(u).$$

Then there exist $z_k \in \mathbb{R}^n$ *and linear transformations* A_k *such that*

$$\det A_k = 1, \quad \textit{for } k \geqslant 1,$$
$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \textit{for } \|x\| = 1,$$
$$1 - C\delta_{k-1} \leqslant \|A_k x\| \leqslant 1 + C\delta_{k-1}, \quad \textit{for } k \geqslant 2,\ \|x\| = 1,$$
$$(1 - \delta_k) B_{\sqrt{2}} \subset \mu^{-1/2} T_k \cdots \mu^{-1/2} T_1 S_{\mu^k}(u) \subset (1 + \delta_k) B_{\sqrt{2}},$$

where $T_k x = A_k(x - z_k)$, $\delta_0 = 1$ *and* $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$ *for* $k \geqslant 1$.

Proof. By Lemma 2.1, there exist z_1 and A_1 with $\det A_1 = 1$ such that

$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \text{for } \|x\| = 1,$$
$$(1 - \delta_1) B_{\sqrt{2}} \subset \mu^{-1/2} T_1 S_\mu(u) \subset (1 + \delta_1) B_{\sqrt{2}}.$$

Let $u_1(x) = \mu^{-1}[u(T_1^{-1}\sqrt{\mu}x) - (\min_\Omega u + \mu)]$ and $\Omega_1 = \sqrt{\mu}^{-1} T_1 S_\mu(u)$. It is easy to verify that

$$1 - \varepsilon_1 \leqslant \det D^2 u_1 \leqslant 1 + \varepsilon_1 \quad \text{in } \Omega_1,$$

and the assumptions in Lemma 2.2 hold. Apply Lemma 2.2 to u_1 in Ω_1, and therefore, there exist z_2 and a linear transformation A_2 with $\det A_2 = 1$ such that

$$1 - C\delta_1 \leqslant \|A_2 x\| \leqslant 1 + C\delta_1, \quad \text{for } \|x\| = 1,$$
$$(1 - \delta_2) B_{\sqrt{2}} \subset \sqrt{\mu}^{-1} T_2 S_\mu(u_1, \Omega_1) = \mu^{-1/2} T_2 \mu^{-1/2} T_1 S_{\mu^2}(u) \subset (1 + \delta_2) B_{\sqrt{2}}.$$

Now use the induction to proceed. Assume that the conclusions in the lemma are valid for the case k. As above, consider the normalized solution and domain given by

$$u_k(x) = \mu^{-k}\Big[u\big(T_1^{-1}\sqrt{\mu}\cdots T_k^{-1}\sqrt{\mu}x\big) - \Big(\min_{\Omega} u + \mu^k\Big)\Big]$$

and $\Omega_k = \mu^{-1/2}T_k\cdots\mu^{-1/2}T_1 S_{\mu^k}(u)$. One can easily check that u_k satisfies

$$1-\varepsilon_k \leqslant \det D^2 u_k \leqslant 1+\varepsilon_k, \quad \text{in } \Omega_k.$$

The induction hypotheses imply that the assumptions in Lemma 2.2 are valid. By applying Lemma 2.2 to u_k in Ω_k, there exist z_{k+1} and a linear transformation A_{k+1} with $\det A_{k+1} = 1$ such that

$$1 - C\delta_k \leqslant \|A_{k+1}x\| \leqslant 1 + C\delta_k, \quad \text{for } \|x\| = 1,$$
$$(1-\delta_{k+1})B_{\sqrt{2}} \subset \sqrt{\mu}^{-1}T_{k+1}S_{\mu}(u_k, \Omega_k) = \mu^{-1/2}T_{k+1}\cdots\mu^{-1/2}T_1 S_{\mu^{k+1}}(u) \subset (1+\delta_{k+1})\mathcal{P}.$$

The proof of Lemma 2.3 is done. □

The following is a refinement of Lemma 2.3.

Lemma 2.4. *Suppose that the assumptions in Lemma* 2.1 *hold and further assume that there is a sequence* $\{\varepsilon_k\}_{k=0}^{\infty}$ *with* $0 < \varepsilon_{k+1} \leqslant \varepsilon_k$ *and* $\varepsilon_0 = \varepsilon \ll \mu$ *such that for* $k \geqslant 1$

$$1-\varepsilon_k \leqslant \det D^2 u \leqslant 1+\varepsilon_k \quad \text{in } S_{\mu^k}(u).$$

Let $u(x_0) = \min_{\Omega} u$. *Then there exist linear transformations* A_k *such that*

$$\det A_k = 1, \quad \text{for } k \geqslant 1,$$
$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \text{for } \|x\| = 1,$$
$$1 - C\delta_{k-1} \leqslant \|A_k x\| \leqslant 1 + C\delta_{k-1}, \quad \text{for } k \geqslant 2,\ \|x\| = 1,$$
$$B_{C^{-1}} \subset \mu^{-k/2}A_k\cdots A_1\big[S_{\mu^k}(u, x_0) - x_0\big] \subset B_3,$$

where $\delta_0 = 1$ *and* $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$ *for* $k \geqslant 1$.

Proof. By Lemma 2.3, there exist $z_k \in \mathbb{R}^n$ and linear transformations A_k such that

$$\det A_k = 1, \quad \text{for } k \geqslant 1,$$
$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \text{for } \|x\| = 1,$$
$$1 - C\delta_{k-1} \leqslant \|A_k x\| \leqslant 1 + C\delta_{k-1}, \quad \text{for } k \geqslant 2,\ \|x\| = 1,$$
$$(1-\delta_k)B_{\sqrt{2}} \subset \mu^{-1/2}T_k\cdots\mu^{-1/2}T_1 S_{\mu^k}(u) \subset (1+\delta_k)B_{\sqrt{2}},$$

where $T_k x = A_k(x - z_k)$, $\delta_0 = 1$ and $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$ for $k \geqslant 1$.

Let $\mathcal{T}_k x = \mu^{-1/2} T_k \cdots \mu^{-1/2} T_1 x$. By induction, we can write $\mathcal{T}_k x = \mu^{-k/2} A_k \cdots A_1 x + y_k$ for some y_k. If we set $y_0 = \mathcal{T}_k x_0$, then $\mathcal{T}_k x = \mu^{-k/2} A_k \cdots A_1 (x - x_0) + y_0$.

As in the proof of Lemma 2.3, let $u_k(y) = \mu^{-k}[u(\mathcal{T}_k^{-1} y) - (\min_\Omega u + \mu^k)]$ and $\Omega_k = \mathcal{T}_k S_{\mu^k}(u)$. Obviously, $u_k(y_0) = \min_{\Omega_k} u_k = -1$, $u_k = 0$ on $\partial\Omega_k$, and u_k satisfies

$$1 - \varepsilon_k \leqslant \det D^2 u_k \leqslant 1 + \varepsilon_k \quad \text{in } \Omega_k.$$

By the Alexandrov estimate, y_0 must lie strictly inside Ω_k. Therefore, there exists $C > 0$ such that

$$B_{C^{-1}}(y_0) \subset \Omega_k \subset (1 + \delta_k) B_{\sqrt{2}} \subset B_3(y_0).$$

The proof of the lemma is completed. □

We now discuss the eccentricity of sections.

Lemma 2.5. *Let u be a convex solution to*

$$\det D^2 u = 1 + g(x) \quad \textit{in } \Omega,$$
$$u = 0 \quad \textit{on } \partial\Omega.$$

Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$ is a convex domain. Let $u(x_0) = \min_\Omega u$. Assume $g(x_0) = 0$. Let $\omega(r)$ denote the modulus of continuity of $g(x)$. Assume that $\omega(2a_2) \ll \mu$ for some small μ and $r^\gamma/\omega(r)$ is almost increasing in $(0, 2a_2]$ for some $0 < \gamma < 1$. Then there exist δ_k decreasing to 0 with $\delta_1 \ll 1$ and linear transforms A_k with $\det A_k = 1$ such that

$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \textit{for } \|x\| = 1, \tag{2.16}$$
$$1 - C\delta_{k-1} \leqslant \|A_k x\| \leqslant 1 + C\delta_{k-1}, \quad \textit{for } k \geqslant 2,\ \|x\| = 1, \tag{2.17}$$

and for $k \geqslant 1$, $\mathcal{T}_k = A_k \cdots A_1$ satisfies

$$C^{-1} e^{-C_\mu \Phi(\sqrt{\mu}^k)} \leqslant \|\mathcal{T}_k x\| \leqslant C e^{C_\mu \Phi(\sqrt{\mu}^k)}, \quad \textit{for } \|x\| = 1, \tag{2.18}$$
$$B_{C^{-1}} \subset \mu^{-k/2} \mathcal{T}_k \big[S_{\mu^k}(u, x_0) - x_0\big] \subset B_C, \tag{2.19}$$

where $\Phi(r) = \int_r^{2a_2} \frac{\omega(t)}{t}\, dt$ and C_μ is a constant dependent on μ.

Proof. Let $\varepsilon_0 = \omega(2a_2)$ and $\varepsilon_k = \operatorname{osc}_{S_{\mu^k}(x_0)}(g)$ for $k \geqslant 1$. By Lemma 2.4, there exist linear transformations A_k with $\det A_k = 1$ such that

$$C^{-1} \leqslant \|A_1 x\| \leqslant C, \quad \text{for } \|x\| = 1,$$
$$1 - C\delta_{k-1} \leqslant \|A_k x\| \leqslant 1 + C\delta_{k-1}, \quad \text{for } k \geqslant 2,\ \|x\| = 1,$$
$$B_{C^{-1}} \subset \mu^{-k/2} A_k \cdots A_1 \big[S_{\mu^k}(u, x_0) - x_0\big] \subset B_3,$$

where $\delta_0 = 1$ and $\delta_k = C(\delta_{k-1}\sqrt{\mu} + \varepsilon_{k-1}/\mu)$ for $k \geqslant 1$.

Obviously, $\delta_1 = C(\sqrt{\mu} + \varepsilon_0/\mu) \ll 1 = \delta_0$ if μ and $\omega(2a_2)/\mu$ are small. Therefore, ε_k and δ_k are decreasing.

To prove the lemma, it suffices to show that $\delta_k \to 0$ and $\mathcal{T}_k = A_k \cdots A_1$ satisfies (2.18). By induction, we have for $k \geqslant 1$

$$\delta_k = \frac{C}{\mu} \sum_{i=0}^{k-1} (C\sqrt{\mu})^{k-1-i} \varepsilon_i + (C\sqrt{\mu})^k. \tag{2.20}$$

Now give a rough estimate for $\mathcal{T}_k^{-1}$. By the estimates of A_k,

$$C^{-1} \prod_{i=1}^{k-1} (1 + C\delta_i)^{-1} \leqslant \|\mathcal{T}_k^{-1}\| \leqslant C \prod_{i=1}^{k-1} (1 - C\delta_i)^{-1}.$$

Therefore, for small $\sigma > 0$, if δ_1 is small enough, then $C^{-1}(1-\sigma)^k \leqslant \|\mathcal{T}_k^{-1}\| \leqslant C(1+\sigma)^k$. It implies from (2.19) that $S_{\mu^k}(u, x_0) - x_0 \subset B_{C_1[(1+\sigma)\sqrt{\mu}]^k}$. Therefore, $\varepsilon_k \leqslant \omega(C_1(1+\sigma)^k \sqrt{\mu}^k)$. Since $r^\gamma/\omega(r)$ is almost increasing, we have

$$\frac{\omega(C_1(1+\sigma)^i \sqrt{\mu}^i)}{[C_1(1+\sigma)^i \sqrt{\mu}^i]^\gamma} \leqslant C_2 \frac{\omega(C_1(1+\sigma)^{k-1} \sqrt{\mu}^{k-1})}{[C_1(1+\sigma)^{k-1} \sqrt{\mu}^{k-1}]^\gamma}, \quad 1 \leqslant i \leqslant k-1.$$

It follows that $\varepsilon_i \leqslant C_2[(1+\sigma)\sqrt{\mu}]^{(i-k+1)\gamma} \omega(C_1(1+\sigma)^{k-1}\sqrt{\mu}^{k-1})$. By (2.20), one obtains for $k \geqslant 2$

$$\begin{aligned}
\delta_k &\leqslant \frac{C_2}{\mu} \sum_{i=1}^{k-1} \big[\big((1+\sigma)\sqrt{\mu}\big)^{-\gamma} C\sqrt{\mu}\big]^{k-1-i} \omega\big(C_1(1+\sigma)^{k-1}\sqrt{\mu}^{k-1}\big) + \left[\frac{\varepsilon_0}{\mu\sqrt{\mu}} + 1\right](C\sqrt{\mu})^k \\
&\leqslant \frac{C_2}{\mu} \omega\big(C_1(1+\sigma)^{k-1}\sqrt{\mu}^{k-1}\big) + \left[\frac{\varepsilon_0}{\mu\sqrt{\mu}} + 1\right](C\sqrt{\mu})^k.
\end{aligned}$$

Since

$$\sum_{i=2}^{k-1} \omega\big(C_1(1+\sigma)^{i-1}\sqrt{\mu}^{i-1}\big) \leqslant \frac{-C_2}{\log[(1+\sigma)\sqrt{\mu}]} \int_{C_1[(1+\sigma)\sqrt{\mu}]^{k-2}}^{2a_2} \frac{\omega(t)}{t}\, dt,$$

it follows that for $k \geqslant 3$

$$\sum_{i=2}^{k-1} \delta_i \leqslant C_\mu \int_{C_1[(1+\sigma)\sqrt{\mu}]^{k-2}}^{2a_2} \frac{\omega(t)}{t}\, dt + C_2\left(\frac{\varepsilon_0}{\sqrt{\mu}} + \mu\right) \leqslant C_\mu \Phi\big(\sqrt{\mu}^{k-2}\big) + C_2\sqrt{\mu},$$

where $C_\mu \leqslant C_2/\mu$. Therefore we have for $k \geqslant 3$

$$C_2^{-1} e^{-C_\mu \Phi(\sqrt{\mu}^{k-2})} \leqslant \prod_{i=2}^{k-1} (1 - C\delta_i) \leqslant \prod_{i=2}^{k-1} (1 + C\delta_i) \leqslant C_2 e^{C_\mu \Phi(\sqrt{\mu}^{k-2})}.$$

This proves (2.18) and Lemma 2.5 follows. $\square$

Theorem 2.1. *Let u be a convex solution to* (1.1) *in Ω and $u=0$ on $\partial\Omega$. Let $B_{\alpha_n}\subset\Omega\subset B_1$ be a convex domain. Assume that there are constants λ, $\Lambda>0$ such that $\lambda\leqslant f(x)\leqslant\Lambda$ in Ω. Furthermore, assume $f\in C(\overline{\Omega})$ with the modulus of continuity φ and $r^\gamma/\varphi(r)$ is almost increasing in* $(0,2]$ *for some $0<\gamma<1$. Let $\Omega_\beta=\{x\in\Omega\colon\ u(x)<(1-\beta)\min_\Omega u\}$ for $0<\beta<1$. For $0<\varepsilon_0\ll\mu\ll1$, let $\tau>0$ such that $\varphi(2\tau)\leqslant\varepsilon_0$. Then there exists a constant $h_0=h_0(\tau,\beta)>0$ such that for $x_0\in\Omega_\beta$, $k\geqslant1$, there exists a linear transform $\mathcal{T}_k$ with $C^{-1}h_0\leqslant\det\mathcal{T}_k\leqslant Ch_0$ satisfying*

$$C_{\tau,\beta}^{-1}e^{-C_\mu\Phi(h_0\sqrt{\mu}^k)}\leqslant\|\mathcal{T}_kx\|\leqslant C_{\tau,\beta}\,e^{C_\mu\Phi(h_0\sqrt{\mu}^k)},\quad \textit{for}\ \ \|x\|=1,\tag{2.21}$$

$$B_{C^{-1}}\subset\mu^{-k/2}\mathcal{T}_k\big[S_{h_0\mu^k}(u,x_0)-x_0\big]\subset B_C,\tag{2.22}$$

where $\Phi(r)=\int_r^2\frac{\varphi(\tau t)}{t}\,dt$ and $C_{\tau,\beta}$ is a constant dependent on τ and β.

Proof. As in the beginning of this section, for $x_0\in\Omega_\beta$ and small $h_0>0$, let E be the Fritz John ellipsoid of $S_{h_0}(u,x_0)$ and T be the affine transformation such that $Tx=A(x-x_0)+y_0$ and $TE=B_1$. Set

$$u^*(y)=\frac{1}{C_T}\big[(u-\ell_{x_0})\big(T^{-1}y\big)-h_0\big],$$

where $C_T=(f(x_0)|\det A|^{-2})^{1/n}$ and $\ell_{x_0}(x)$ is the supporting affine function of u at x_0. It is easy to verify that $u^*(y)$ satisfies

$$\det D^2u^*=f^*(y)\quad\text{in } S^*=TS_{h_0}(x_0),$$

where $f^*(y)=f(T^{-1}y)/f(x_0)$. Moreover, we have

$$\big|f^*(y_1)-f^*(y_2)\big|\leqslant\frac{1}{f(x_0)}\varphi\big(\big\|A^{-1}\big\|\cdot|y_1-y_2|\big).$$

Note that $\|A^{-1}\|\to0$, as $h_0\to0$, by [1] or [8]. Choose h_0 such that $\|A^{-1}\|\leqslant\tau$. By Lemma 2.5, there exists a linear transform $\mathcal{T}_k$ with $\det\mathcal{T}_k=1$ such that

$$C^{-1}e^{-C_\mu\Phi(\sqrt{\mu}^k)}\leqslant\|\mathcal{T}_ky\|\leqslant Ce^{C_\mu\Phi(\sqrt{\mu}^k)},\quad\text{for}\ \ \|y\|=1,$$
$$B_{C^{-1}}\subset\mu^{-k/2}\mathcal{T}_k\big[S_{\mu^k}(u^*,y_0)-y_0\big]\subset B_C,$$

where $\Phi(r)=\int_r^2\frac{\varphi(\tau t)}{t}\,dt$ and C_μ is a constant dependent on μ. By (2.2), $C_1\leqslant C_T/h_0\leqslant C_2$. If $\theta=C_1h_0/C_T$, then $C_1/C_2\leqslant\theta\leqslant1$. Let $S_1^*=\mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(u^*,y_0)-y_0]$ and $u_1^*(z)=\mu^{-k}[u^*(y_0+\mu^{k/2}\mathcal{T}_k^{-1}z)-u^*(y_0)-\mu^k]$. We can easily verify that

$$\det D^2u_1^*=f^*\big(y_0+\mu^{k/2}\mathcal{T}_k^{-1}z\big)\quad\text{in } S_1^*.$$

By the Alexandrov estimate, we have $B_{C_1}\subset S_\theta(u_1^*,0)\subset B_C$. Since

$$S_\theta\big(u_1^*,0\big)=\mu^{-k/2}\mathcal{T}_k\big[S_{\theta\mu^k}(u^*,y_0)-y_0\big]=\mu^{-k/2}\mathcal{T}_kA\big[S_{C_T\theta\mu^k}(u,x_0)-x_0\big],$$

$\hat{T}_k=\mathcal{T}_kA$ and $\hat{h}_0=C_T\theta=C_1h_0$ are the transform and constant in Theorem 2.1 we seek. $\square$

3. Estimates of the mean oscillation

In this section, we establish estimates for the mean oscillation of second derivatives of solutions to (1.1).

We firstly prove the following estimates for normalized solutions.

Theorem 3.1. *Let u be the convex solution to*

$$\det D^2 u = 1 + g(x) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

Assume that $B_{a_1} \subset \Omega \subset B_{a_2}$ is a convex domain. Let $u(x_0) = \min_\Omega u$. Assume $g(x_0) = 0$. Let $\omega(r)$ denote the modulus of continuity of $g(x)$. Assume that $\omega(2a_2) \ll \mu$ for some small μ and $r^\gamma/\omega(r)$ is almost increasing in $(0, 2a_2]$ for some $0 < \gamma < 1$. Then there exist $0 < \delta < 1$ and small $r_0 > 0$ such that for $0 < r \leqslant r_0$

$$\inf_{M \in \mathcal{S}} \fint_{B_r(x_0)} \left|D^2 u - M\right|^\delta \leqslant C\psi^\delta(r)\left[1 + \int_\Omega \left|D^2 u\right|^\delta\right],$$

where $\psi(r) = e^{C_1\Phi(r)}\omega(e^{C_1\Phi(r)}r)$, $\Phi(r) = \int_r^{2a_2} \frac{\omega(t)}{t}\,dt$, and $\mathcal{S}$ denotes the space of real $n \times n$ symmetric matrices.

Proof. By Lemma 2.5, there exist linear transforms A_k and $\mathcal{T}_k$ satisfying (2.16)–(2.19). Let $u^*(y) = \mu^{-k}[u(x_0 + \mu^{k/2}\mathcal{T}_k^{-1}y) - (\min_\Omega u + \mu^k)]$ and $S^* = \mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(u, x_0) - x_0]$. Obviously, we have

$$\det D^2 u^* = 1 + g^*(y) \quad \text{in } S^*,$$

and $u^* = 0$ on ∂S^*, where $g^*(y) = g(x_0 + \mu^{k/2}\mathcal{T}_k^{-1}y)$. By (2.18) and (2.19)

$$\operatorname{osc}_{S^*} g^*(y) = \operatorname{osc}_{S_{\mu^k}(u,x_0)} g(x) \leqslant \omega\bigl(Ce^{C_\mu\Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k\bigr),$$

where $\Phi(r) = \int_r^{2a_2} \frac{\omega(t)}{t}\,dt$.

Let w be the solution to the following problem

$$\begin{cases} \det D^2 w = 1 & \text{in } S^*, \\ w = 0 & \text{on } \partial S^*. \end{cases}$$

By the comparison principle, $(1 + \operatorname{osc}_{S^*} g^*)w \leqslant u^* \leqslant (1 - \operatorname{osc}_{S^*} g^*)w$ in S^*. Therefore

$$\left|u^*(y) - w(y)\right| \leqslant C_1 \operatorname{osc}_{S^*} g^* \leqslant C_1\omega\bigl(Ce^{C_\mu\Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k\bigr), \quad \text{in } S^*. \tag{3.1}$$

Now we claim that w satisfies the following Campanato inequality

$$\inf_{M \in \mathcal{S}} \int_{B_\rho} \left|D^2 w - M\right|^p \leqslant C_p\left(\frac{\rho}{R}\right)^{n+p} \inf_{M \in \mathcal{S}} \int_{B_R} \left|D^2 w - M\right|^p, \tag{3.2}$$

for $p > 0$, $0 < \rho < R \leqslant R_0 = (2C)^{-1}$, where C is the constant in (2.19) and $\mathcal{S}$ is the space of real $n \times n$ symmetric matrices.

To prove (3.2), since $B_{C^{-1}} = B_{2R_0} \subset S^*$, by standard estimates of the Monge–Ampère equation,

$$\|w\|_{C^3(\overline{B}_{R_0})} \leqslant C_1$$

and $D^2 w$ is strictly positively definite. Let $B_{r_0}(z_0) \subset B_{R_0}$. By differentiating the equation $\det D^2 w = 1$, it is easy to check that if $\hat{w}(x) = w(x) - [w(z_0) + Dw(z_0)(x - z_0) + \frac{1}{2}(x - z_0)^t D^2 w(z_0)(x - z_0)]$ then $D_k \hat{w}$ satisfies the uniformly elliptic equation

$$\operatorname{tr}\big(\big(D^2 w\big)^{-1} D^2(D_k \hat{w})\big) = 0, \quad \text{in } B_{r_0}(z_0).$$

By the Schauder estimates, $\|D^2 D_k \hat{w}\|_{L^\infty(B_{r_0/2}(z_0))} \leqslant (C_1/r_0^2)\|D_k \hat{w}\|_{L^\infty(B_{r_0}(z_0))}$. It follows that for $0 < r < r_0/2$

$$\operatorname{osc}_{B_r(z_0)} D^2 w \leqslant C_1 r \big\|D^3 w\big\|_{L^\infty(B_{r_0/2}(z_0))} \leqslant C_1 \left(\frac{r}{r_0}\right) \operatorname{osc}_{B_{r_0}(z_0)} D^2 w.$$

Similar to the proof of [4, Theorem 3.2], (3.2) is obtained.

Let $v = u^* - w$. From (3.2), for $0 < \rho \leqslant R_0$, $0 < p \leqslant 1$

$$\begin{aligned}\inf_{M \in \mathcal{S}} \int_{B_\rho} \big|D^2 u^* - M\big|^p &\leqslant C_p \rho^{n+p} \inf_{M \in \mathcal{S}} \int_{B_{R_0}} \big|D^2 w - M\big|^p + \inf_{M \in \mathcal{S}} \int_{B_\rho} \big|D^2 v - M\big|^p \\ &\leqslant C_p \rho^{n+p} \inf_{M \in \mathcal{S}} \int_{B_{R_0}} \big|D^2 u^* - M\big|^p + C_p \int_{B_{R_0}} \big|D^2 v\big|^p. \qquad (3.3)\end{aligned}$$

We need to estimate the integral of $|D^2 v|^p$. Let $\mathcal{M}(D^2 w) = (\det D^2 w)^{1/n}$. Let $L_w v = \frac{1}{n}(\det D^2 w)^{1/n} \operatorname{tr}((D^2 w)^{-1} D^2 v)$ be the linearized operator of the operator $\mathcal{M}$. Since $\mathcal{M}$ is a concave operator, we obtain

$$L_{u^*}(u^* - w) \leqslant \mathcal{M}\big(D^2 u^*\big) - \mathcal{M}\big(D^2 w\big) = (1 + g^*)^{1/n} - 1 \leqslant L_w(u^* - w), \quad \text{in } S^*.$$

Since $L_w v \geqslant -C_1 \operatorname{osc}_{S^*} g^*$ and L_w is uniformly elliptic in $B_{3R_0/2}$, by one-sided $W^{2,\delta}$ estimates in [3, Lemma 7.8] and (3.1), there exists $0 < \delta_1 < 1$ such that for $1 \leqslant i, j \leqslant n$

$$\bigg(\fint_{B_{R_0}} \big|(D_{ij} v)^+\big|^{\delta_1}\bigg)^{1/\delta_1} \leqslant C_1\big(\|v\|_{L^\infty(S^*)} + \operatorname{osc}_{S^*} g^*\big) \leqslant C_1 \operatorname{osc}_{S^*} g^*. \qquad (3.4)$$

On the other hand, $L_{u^*} v \leqslant C_1 \operatorname{osc}_{S^*} g^*$ and hence

$$\hat{L}_{u^*} v = \operatorname{tr}\big(\det D^2 u^* \big(D^2 u^*\big)^{-1} D^2 v\big) \leqslant C_1 \operatorname{osc}_{S^*} g^*, \quad \text{in } S^*.$$

Recently, [9] established $W^{2,\delta}$ estimates with small δ for the equation $\hat{L}_{u^*}v = g$ which is not uniformly elliptic in general. A straightforward modification of their proof gives rise to one-sided $W^{2,\delta}$ estimates for supersolutions. More precisely, there exists $0 < \delta_2 < 1$ such that for $1 \leqslant i, j \leqslant n$

$$\left(\int_{B_{R_0}} \left|(D_{ij}v)^-\right|^{\delta_2} \right)^{1/\delta_2} \leqslant C_1\big(\|v\|_{L^\infty(S^*)} + \operatorname{osc}_{S^*} g^*\big) \leqslant C_1 \operatorname{osc}_{S^*} g^*. \tag{3.5}$$

Choose $p = \delta = \min\{\delta_1, \delta_2\}$ in (3.3). By (3.3)–(3.5), we obtain for $0 < \rho \leqslant R_0$

$$\inf_{M\in\mathcal{S}} \int_{B_\rho} \left|D^2u^* - M\right|^\delta \leqslant C_1\rho^{n+\delta} \inf_{M\in\mathcal{S}} \int_{B_{R_0}} \left|D^2u^* - M\right|^\delta + C_1\omega^\delta\big(Ce^{C_\mu \Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k\big).$$

By Lemma 2.5, $B_{C_1^{-1}\sqrt{\mu}} \subset S_\mu(u^*, 0) \subset B_{C_1\sqrt{\mu}}$. Therefore

$$\inf_{M\in\mathcal{S}} \int_{S_\mu(u^*,0)} \left|D^2u^* - M\right|^\delta \leqslant C_1\sqrt{\mu}^{n+\delta} \inf_{M\in\mathcal{S}} \int_{S^*} \left|D^2u^* - M\right|^\delta + C_1\omega^\delta\big(Ce^{C_\mu \Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k\big).$$

Recall $S^* = \mu^{-k/2}\mathcal{T}_k[S_{\mu^k}(u, x_0) - x_0]$, and note $S_\mu(u^*, 0) = \mu^{-k/2}\mathcal{T}_k[S_{\mu^{k+1}}(u, x_0) - x_0]$. By the change of variables, it follows that

$$\begin{aligned} &\inf_{M\in\mathcal{S}} \int_{S_{\mu^{k+1}}(u,x_0)} \left|\big(\mathcal{T}_k^{-1}\big)^t\big(D^2u - M\big)\mathcal{T}_k^{-1}\right|^\delta \\ &\quad \leqslant C_1\sqrt{\mu}^{n+\delta} \inf_{M\in\mathcal{S}} \int_{S_{\mu^k}(u,x_0)} \left|\big(\mathcal{T}_k^{-1}\big)^t\big(D^2u - M\big)\mathcal{T}_k^{-1}\right|^\delta \\ &\qquad + C_1\sqrt{\mu}^{kn}\omega^\delta\big(Ce^{C_\mu \Phi(\sqrt{\mu}^k)}\sqrt{\mu}^k\big). \end{aligned} \tag{3.6}$$

To simplify notations, let $\bar{\mu} = \sqrt{\mu}$ and

$$b_k = \inf_{M\in\mathcal{S}} \int_{S_{\mu^k}(u,x_0)} \left|\big(\mathcal{T}_k^{-1}\big)^t\big(D^2u - M\big)\mathcal{T}_k^{-1}\right|^\delta.$$

Since $\mathcal{T}_{k+1} = A_{k+1}\mathcal{T}_k$, by (3.6) and (2.17), for $k \geqslant 1$

$$\begin{aligned} b_{k+1} &\leqslant \left\|A_{k+1}^{-1}\right\|^{2\delta} \inf_{M\in\mathcal{S}} \int_{S_{\mu^{k+1}}(u,x_0)} \left|\big(\mathcal{T}_k^{-1}\big)^t\big(D^2u - M\big)\mathcal{T}_k^{-1}\right|^\delta \\ &\leqslant C_1\bar{\mu}^{n+\delta}b_k + C_2\bar{\mu}^{kn}\omega^\delta\big(Ce^{C_\mu \Phi(\bar{\mu}^k)}\bar{\mu}^k\big). \end{aligned}$$

Let $F(t) = t^n \omega^\delta(Ce^{C_\mu \Phi(t)}t)$. For $\gamma < \beta < 1$, choose $\bar{\mu}$ small enough such that $C_1 \bar{\mu}^{\delta(1-\beta)} < 1$. Therefore

$$b_{k+1} \leqslant \bar{\mu}^{n+\delta\beta} b_k + C_2 F\big(\bar{\mu}^k\big), \quad k \geqslant 1. \tag{3.7}$$

By the induction, from (3.7), we obtain

$$b_{k+1} \leqslant \bar{\mu}^{k(n+\delta\beta)} b_1 + C_2 \sum_{i=1}^{k} \bar{\mu}^{(k-i)(n+\delta\beta)} F\big(\bar{\mu}^i\big). \tag{3.8}$$

Let $h(t) = \omega(Ce^{C_\mu \Phi(t)}t)$. We now show $t^\gamma / h(t)$ is almost increasing if $\omega(2a_2) \ll \mu$. Note that

$$\frac{t^\gamma}{h(t)} = \frac{[Ce^{C_\mu \Phi(t)}t]^\gamma}{\omega(Ce^{C_\mu \Phi(t)}t)} \cdot \big[Ce^{C_\mu \Phi(t)}\big]^{-\gamma}.$$

Obviously, $e^{-\gamma C_\mu \Phi(t)}$ is increasing. Since $t^\gamma / \omega(t)$ is almost increasing by the assumptions, and

$$\frac{d}{dt}\big(e^{C_\mu \Phi(t)}t\big) = e^{C_\mu \Phi(t)}\big(1 - C_\mu \omega(t)\big) \geqslant e^{C_\mu \Phi(t)}\big(1 - C_\mu \omega(2a_2)\big) > 0,$$

we conclude that $t^\gamma / h(t)$ is almost increasing.

For $1 \leqslant i \leqslant k$, we have the following

$$\begin{aligned}
\bar{\mu}^{(k-i)(n+\delta\beta)} F\big(\bar{\mu}^i\big) &= \bar{\mu}^{(k-i)(n+\delta\beta)} \bar{\mu}^{i(n+\delta\gamma)} \left[\frac{h(\bar{\mu}^i)}{\bar{\mu}^{i\gamma}}\right]^\delta \\
&\leqslant C_2 \bar{\mu}^{(k-i)(n+\delta\beta)} \bar{\mu}^{i(n+\delta\gamma)} \left[\frac{h(\bar{\mu}^k)}{\bar{\mu}^{k\gamma}}\right]^\delta \\
&= C_2 \bar{\mu}^{kn} h^\delta\big(\bar{\mu}^k\big) \bar{\mu}^{(k-i)\delta(\beta-\gamma)}.
\end{aligned}$$

From (3.8) and since $t^\beta / h(t)$ is bounded, it follows that for $k \geqslant 1$

$$\begin{aligned}
b_{k+1} &\leqslant \bar{\mu}^{k(n+\delta\beta)} b_1 + C_2 \bar{\mu}^{kn} h^\delta\big(\bar{\mu}^k\big) \sum_{i=1}^{k} \bar{\mu}^{(k-i)\delta(\beta-\gamma)} \\
&\leqslant C_2 \big(\bar{\mu}^k\big)^n \big[h\big(\bar{\mu}^k\big)\big]^\delta (1 + b_1).
\end{aligned}$$

Since $t^\gamma / h(t)$ is almost increasing, we have $b_k \leqslant \overline{C}_\mu (\bar{\mu}^k)^n [h(\bar{\mu}^k)]^\delta (1 + b_1)$, for $k \geqslant 2$. Recalling $\bar{\mu} = \sqrt{\mu}$ and the definition of b_k and $h(t)$, by (2.18), one can easily verify that for $k \geqslant 2$

$$\begin{aligned}
\inf_{M \in \mathcal{S}} \int\limits_{S_{\mu^k}(u,x_0)} \big|D^2 u - M\big|^\delta &\leqslant C_2 \|\mathcal{T}_k\|^{2\delta} b_k \\
&\leqslant \overline{C}_\mu \big(\sqrt{\mu}^k\big)^n \big[e^{C_\mu \Phi(\sqrt{\mu}^k)} \omega\big(Ce^{C_\mu \Phi(\sqrt{\mu}^k)} \sqrt{\mu}^k\big)\big]^\delta (1 + b_1).
\end{aligned} \tag{3.9}$$

Now fix μ small. We claim that for $0 < t \leqslant a_2$ and $0 < \theta < 1$

$$\Phi(t) \leqslant \Phi(\theta t) \leqslant C_\theta \Phi(t).$$

Indeed, we have

$$\Phi(\theta t) \leqslant \Phi(t) + \log(1/\theta)\omega(t) \leqslant \Phi(t) + \frac{\log(1/\theta)}{\log 2} \int_t^{2t} \frac{\omega(r)}{r}\, dr \leqslant \left(1 + \frac{\log(1/\theta)}{\log 2}\right)\Phi(t).$$

Similarly, one can prove $\Phi(\sqrt{\rho}) \leqslant \Phi(\rho) \leqslant (2 + |\log(2a_2)|/\log 2)\Phi(\sqrt{\rho})$, for $0 < \rho \leqslant \min\{1, a_2^2\}$.

By properties of Φ, from (3.9) and (2.18)–(2.19), the standard argument yields for $0 < \rho \leqslant \mu^2$

$$\inf_{M \in \mathcal{S}} \int_{S_\rho(u,x_0)} \left|D^2 u - M\right|^\delta \leqslant C_2 \bar{\rho}^n \left[e^{C_3 \Phi(\bar{\rho})} \omega\left(C e^{C_3 \Phi(\bar{\rho})} \bar{\rho}\right)\right]^\delta (1 + b_1), \tag{3.10}$$

where $\bar{\rho} = \sqrt{\rho}$, and for $0 < \rho \leqslant \mu$

$$B_{C_2^{-1} \bar{\rho} e^{-C_3 \Phi(\bar{\rho})}}(x_0) \subset S_\rho(u, x_0) \subset B_{C_2 \bar{\rho} e^{C_3 \Phi(\bar{\rho})}}(x_0), \tag{3.11}$$

where $\bar{\rho} = \sqrt{\rho}$.

Let $R = (C_2)^2 \rho e^{2C_3 \Phi(\rho)}$. Then $C_2^{-1} \sqrt{R} e^{-C_3 \Phi(\sqrt{R})} \geqslant \sqrt{\rho} e^{C_3 \Phi(\rho)} e^{-C_3 \Phi(\rho)} \geqslant \sqrt{\rho}$. By (3.11), it implies that $B_{\sqrt{\rho}}(x_0) \subset S_R(u, x_0)$. Moreover, if ρ is small, then $\rho \leqslant R \leqslant \sqrt{\rho}$. Indeed,

$$\Phi(\rho) \leqslant \omega(2a_2)\left[\log(2a_2) + \log(1/\rho)\right] \quad \text{and} \quad R \leqslant C_4 \rho e^{\frac{1}{3} \log(1/\rho)} \leqslant \sqrt{\rho},$$

since ρ and $\omega(2a_2)$ are small. Then from (3.10), for small ρ

$$\inf_{M \in \mathcal{S}} \int_{B_{\sqrt{\rho}}(x_0)} \left|D^2 u - M\right|^\delta \leqslant C_2 \sqrt{R}^n \left[e^{C_3 \Phi(\sqrt{R})} \omega\left(C e^{C_3 \Phi(\sqrt{R})} \sqrt{R}\right)\right]^\delta (1 + b_1).$$

Noting that $\Phi(t)$ is decreasing and $\Phi(\rho) \leqslant C\Phi(\sqrt{\rho})$, we obtain

$$\inf_{M \in \mathcal{S}} \int_{B_{\sqrt{\rho}}(x_0)} \left|D^2 u - M\right|^\delta \leqslant C_2 \sqrt{\rho}^n e^{C_5 \Phi(\sqrt{\rho})} \omega^\delta\left(e^{C_5 \Phi(\sqrt{\rho})} \sqrt{\rho}\right)(1 + b_1).$$

The proof of Theorem 3.1 is completed. □

Proof of Theorem A. Similar to the proof of Theorem 2.1, for $x_0 \in \Omega_\beta = \{x \in \Omega\colon u(x) < (1 - \beta)\min_\Omega u\}$, $0 < \beta < 1$, and small $h_0 > 0$, let E be the Fritz John ellipsoid of $S_{h_0}(u, x_0)$ and T be the affine transformation given by $Tx = A(x - x_0) + y_0$ such that $TE = B_1$. Set

$$u^*(y) = \frac{1}{C_T}\left[(u - \ell_{x_0})\left(T^{-1} y\right) - h_0\right],$$

where $C_T = (f(x_0)|\det A|^{-2})^{1/n}$ and $\ell_{x_0}(x)$ is the supporting affine function of u at x_0. Then

$$\det D^2 u^* = f^*(y) \quad \text{in } S^* = T S_{h_0}(x_0),$$

where $f^*(y) = f(T^{-1}y)/f(x_0)$. Choose $h_0 = h_0(\tau, \beta)$ such that $\|A^{-1}\| \leqslant \tau$ and $S_{h_0}(x_0) \subset \Omega_{(1+\beta)/2}$. Obviously,

$$\left| f^*(y_1) - f^*(y_2) \right| \leqslant \frac{1}{f(x_0)} \varphi\bigl(\tau |y_1 - y_2|\bigr).$$

Applying Theorem 3.1 to u^*, one obtains for $0 < r \leqslant r_0$

$$\inf_{M \in \mathcal{S}} \int\limits_{B_r(y_0)} \left| D^2 u^* - M \right|^\delta \leqslant C r^n \psi^\delta(r) \left[1 + \int\limits_{S^*} \left| D^2 u^* \right|^\delta \right],$$

where $\psi(r) = e^{C_1 \Phi(r)} \varphi(e^{C_1 \Phi(r)} r)$, and $\Phi(r) = \int_r^2 \frac{\varphi(\tau t)}{t}\, dt$. Obviously, $T B_{r/\|A\|}(x_0) \subset B_r(y_0)$. By the change of variables, we have

$$\inf_{M \in \mathcal{S}} \int\limits_{B_{r/\|A\|}(x_0)} \left| D^2 u - M \right|^\delta \leqslant C_{\tau,\beta} r^n \psi^\delta(r) \left[1 + \int\limits_{S_{h_0}(x_0)} \left| D^2 u \right|^\delta \right],$$

where $C_{\tau,\beta}$ depends on τ and β. Let $\rho = r/\|A\|$. Note that $\|A\|$ is large since T dilates the small ellipsoid E onto B_1. Since $\Phi(r)$ and $\varphi(\tau r)/r^\gamma$ are almost decreasing, for $0 < \rho \leqslant r_0/\|A\|$

$$\inf_{M \in \mathcal{S}} \int\limits_{B_\rho(x_0)} \left| D^2 u - M \right|^\delta \leqslant C_{\tau,\beta} \rho^n \psi^\delta(\rho) \left[1 + \int\limits_{\Omega_{(1+\beta)/2}} \left| D^2 u \right|^\delta \right].$$

By [4, Corollary 2.3], we conclude that $D^2 u \in \mathrm{BMO}_\psi(\Omega_\beta)$ and

$$\inf_{M \in \mathcal{S}} \fint\limits_{B_\rho(x_0)} \left| D^2 u - M \right| \leqslant C_{\tau,\beta} \psi(\rho) \left[1 + \left(\int\limits_{\Omega_{(1+\beta)/2}} \left| D^2 u \right|^\delta \right)^{1/\delta} \right].$$

Note that the L^p estimates of $D^2 u$ were established in [2]. Therefore, the proof of Theorem A(i) is done.

We finally prove that if $\lim_{r \to 0} \frac{\varphi(r)}{(\log(1/r))^{-1}} = 0$, then $D^2 u \in \mathrm{BMO}_{\log^{-(1-\varepsilon)}(1/r)}$ for any $0 < \varepsilon < 1$.

To prove, choose τ small enough such that $\varphi(\tau r) \leqslant \varepsilon (\log(1/\tau r))^{-1}$ for $r \leqslant 2$. For small r, it is easy to verify $\Phi(r) \leqslant \varepsilon \log(\log(1/r))$ and $r \leqslant e^{C_1 \Phi(r)} r \leqslant \sqrt{r}$. Therefore, $\psi(r) \leqslant 2 \log^{-(1 - C_1 \varepsilon)}(1/r)$. Thus, we complete the proof. $\square$

Remark 3.2. Under the assumptions of Theorem A(i), if we furthermore assume that $\frac{\log^{-2}(1/r)}{\varphi(r)}$ is almost increasing, then $D^2 u \in \mathrm{BMO}_\psi$ with $\psi(r) = e^{C \Phi(r)} \varphi(r)$.

To prove, by choosing small $\tau > 0$, one can obtain $r \leqslant e^{C\Phi(r)} r \leqslant \sqrt{r}$ for small $r > 0$. The fact that $\frac{\log^{-2}(1/r)}{\varphi(r)}$ is almost increasing implies that $\varphi(\sqrt{r}) \leqslant 4K\varphi(r)$ for some constant K. Therefore, $\varphi(e^{C\Phi(r)} r) \leqslant \varphi(\sqrt{r}) \leqslant 4K\varphi(r)$.

References

[1] L. Caffarelli, A localization property of viscosity solutions to the Monge–Ampère equation and their strict convexity, Ann. of Math. 131 (1990) 129–134.
[2] L. Caffarelli, Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation, Ann. of Math. 131 (1990) 135–150.
[3] L. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., Providence, RI, 1995.
[4] L. Caffarelli, Q. Huang, Estimates in the generalized Campanato–John–Nirenberg spaces for fully nonlinear elliptic equations, Duke Math. J. 118 (2003) 1–17.
[5] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations I. Monge–Ampère equation, Comm. Pure Appl. Math. 37 (1984) 369–402.
[6] S.-Y. Cheng, S.-T. Yau, On the regularity of the Monge–Ampère equation $\det(\partial^2 u/\partial x_i \partial x_j) = F(x, u)$, Comm. Pure. Appl. Math. 30 (1977) 41–68.
[7] C.E. Gutiérrez, The Monge–Ampère Equation, Progress Nonlinear Differential Equations Appl., vol. 44, Birkhäuser, Boston, 2001.
[8] C.E. Gutiérrez, Q. Huang, Geometric properties of the sections of solutions to the Monge–Ampère equation, Trans. Amer. Math. Soc. 352 (2000) 4381–4396.
[9] C.E. Gutiérrez, F. Tournier, $W^{2,p}$ estimates for the linearized Monge–Ampère equation, preprint.
[10] J. Kovats, Fully nonlinear elliptic equations and the Dini condition, Comm. Partial Differential Equations 22 (1997) 1911–1927.
[11] N.V. Krylov, Bounded nonhomogeneous elliptic and parabolic equations, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982) 487–523.
[12] N.V. Krylov, Bounded nonhomogeneous elliptic and parabolic equations in a domain, Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983) 75–108.
[13] N.V. Krylov, On degenerate nonlinear elliptic equations, Mat. Sb. 120 (162) (1983) 311–330.
[14] A.V. Pogorelov, On the regularity of generalized solutions of the equation $\det(\partial^2 u/\partial x_i \partial x_j) = \varphi(x_1, \dots, x_n) > 0$, Soviet Math. Dokl. 12 (5) (1971) 1436–1440.
[15] J. Urbas, Regularity of generalized solutions of the Monge–Ampère equation, Math. Z. 197 (1988) 365–393.
[16] X.-J. Wang, Remarks on the regularity of Monge–Ampère equations, in: Proceedings of the International Conference on Nonlinear PDEs, Hangzhou, 1992, Academic Press, Beijing, 1992.

Further reading

[17] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer, 1983.
[18] F. John, Quasi-isometric mappings, Semineri 1962–1963 di Analisi Algebra, Geometria E Topologia, vol. II, Rome, 1965.
[19] A.V. Pogorelov, The Minkowski Multi-Dimensional Problem, Wiley, New York, 1978.
[20] J.-O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, Indiana Univ. Math. J. 28 (1979) 511–544.